

# Recent results in the BFKL theory<sup>★</sup>

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The Balitsky-Fadin-Kuraev-Lipatov (BFKL) approach for the cross sections at high energy  $\sqrt{s}$  in perturbative QCD is briefly reviewed. The role of gluon Reggeization in the derivation of the BFKL equation and its compatibility with  $s$ -channel unitarity (“bootstrap”) are discussed.

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## 1 Gluon Reggeization in perturbative QCD

The key role in the derivation of the BFKL equation [1] for the processes at high energy  $\sqrt{s}$  in perturbative QCD is played by the gluon Reggeization. “Reggeization” of a given elementary particle usually means that the amplitude of a scattering process with exchange of the quantum numbers of that particle in the  $t$ -channel goes like  $s^{j(t)}$  in the Regge limit  $s \gg |t|$ . The function  $j(t)$  is called “Regge trajectory” of the given particle and takes the value of the spin of that particle when  $t$  is equal to its squared mass. In perturbative QCD, the notion of gluon Reggeization is used in a stronger sense. It means not only that a Reggeon exists with the quantum numbers of the gluon, negative signature and with a trajectory  $j(t) = 1 + \omega(t)$  passing through 1 at  $t = 0$ , but also that this Reggeon gives the leading contribution in each order of perturbation theory to the amplitude of processes with large  $s$  and fixed (i.e. not growing with  $s$ ) squared momentum transfer  $t$ .

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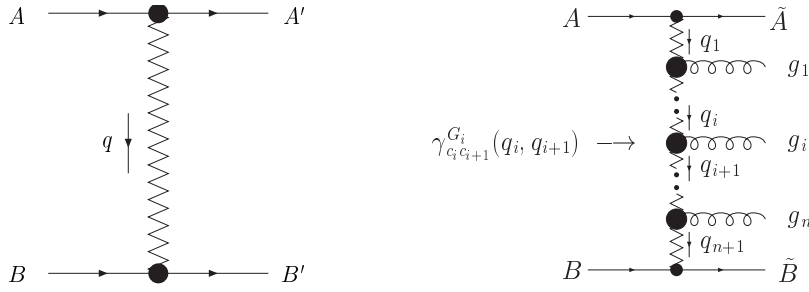


Fig. 1. (Left) Diagrammatical representation of  $(A_8^-)^{A'B'}_{AB}$ . The zig-zag line is the Reggeized gluon, the black blobs the PPR effective vertices.

Fig. 2. (Right) Diagrammatical representation of the production amplitude  $A_{AB}^{\tilde{A}\tilde{B}+n}$  in the LLA.

To be definite, let us consider the elastic process  $A + B \longrightarrow A' + B'$  with exchange of gluon quantum numbers in the  $t$ -channel, i.e. for octet color representation in the  $t$ -channel and negative signature (see Fig. 1). Gluon Reggeization means that, in the Regge kinematical region  $s \simeq -u \rightarrow \infty$ ,  $t$  fixed (i.e. not growing with  $s$ ), the amplitude of this process takes the form

$$(A_8^-)^{A'B'}_{AB} = \Gamma_{A'A}^c \left[ \left( \frac{-s}{-t} \right)^{j(t)} - \left( \frac{s}{-t} \right)^{j(t)} \right] \Gamma_{B'B}^c. \quad (1)$$

Here  $c$  is a color index and  $\Gamma_{P'P}^c$  are the particle-particle-Reggeon (PPR) vertices, not depending on  $s$ . They can be written as  $\Gamma_{P'P}^c = g \langle P' | T^c | P \rangle \Gamma_{P'P}$ , where  $g$  is the QCD coupling constant and  $T^c$  are the color group generators in the fundamental (adjoint) representation for quarks (gluons). This form of the amplitude has been proved rigorously [2] to all orders of perturbation theory in the leading-logarithmic approximation (LLA), which means resummation of the terms  $\alpha_s^n \ln^n s$ . In this approximation  $\Gamma_{P'P}$  is given simply by  $\delta_{\lambda_{P'} \lambda_P}$ , where  $\lambda_P$  is the helicity of the particle  $P$ , and the (deviation from 1 of the) Reggeized gluon trajectory enters with 1-loop accuracy [3],

$$\omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{D-1}} \frac{N}{2} \int \frac{d^{D-2} k_\perp}{k_\perp^2 (q - k)_\perp^2} = -\frac{g^2 N \Gamma(1-\epsilon)}{(4\pi)^{D/2}} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (-q_\perp^2)^\epsilon.$$

Here  $D = 4 + 2\epsilon$  has been introduced in order to regularize the infrared divergences and the integration is performed in the space transverse to the momenta of the initial colliding particles<sup>1</sup>. In the NLA, which means resummation of the terms  $\alpha_s^{n+1} \ln^n s$ , the form (1) has been checked in the first three orders of perturbation theory [4] and is only assumed to be valid to all orders. In this approximation, the NLA contribution to the PPR vertices takes the

<sup>1</sup> In the following, since the transverse component of any momentum is obviously space-like, the notation  $p_\perp^2 = -\vec{p}^2$  will be also used.

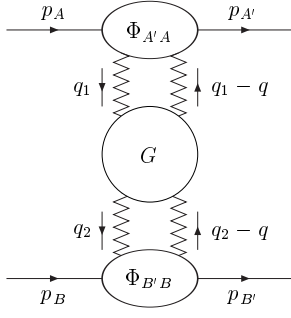


Fig. 3. (Left) Diagrammatic representation of  $A_{AB}^{A'B'}$  (for a definite color group representation) as derived from  $s$ -channel unitarity. The ovals are the impact factors of the particles  $A$  and  $B$ , the circle is the Green's function for the Reggeon-Reggeon scattering.

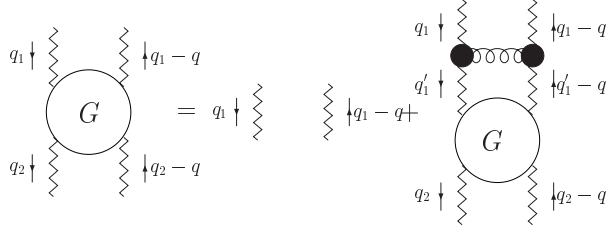


Fig. 4. (Right) Schematic representation of the BFKL equation in the LLA.

form  $\Gamma_{P'P} = \delta_{\lambda_{P'}\lambda_P} \Gamma_{PP}^{(+)} + \delta_{\lambda_{P'}, -\lambda_P} \Gamma_{PP}^{(-)}$ , where a helicity non-conserving term appeared, and the Reggeized gluon trajectory enters with 2-loop accuracy.

## 2 BFKL in the LLA

Amplitudes with quantum numbers in the  $t$ -channel different from the gluon ones are obtained in the BFKL approach by means of unitarity relations, thus calling for inelastic amplitudes. In the LLA, the main contributions to the unitarity relations from inelastic amplitudes come from the multi-Regge kinematics, i.e. when rapidities of the produced particles are strongly ordered and their transverse momenta do not grow with  $s$ . In the multi-Regge kinematics, the real part<sup>2</sup> of the production amplitudes takes a simple factorized form, due to gluon Reggeization,

$$A_{AB}^{\tilde{A}\tilde{B}+n} = 2s \Gamma_{\tilde{A}\tilde{A}}^{c_1} \left( \prod_{i=1}^n \gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1}) \left( \frac{s_i}{s_R} \right)^{\omega_i} \frac{1}{t_i} \right) \frac{1}{t_{n+1}} \left( \frac{s_{n+1}}{s_R} \right)^{\omega_{n+1}} \Gamma_{\tilde{B}\tilde{B}}^{c_{n+1}}, \quad (2)$$

where  $s_R$  is an energy scale, irrelevant in the LLA,  $\gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1})$  is the (non-local) effective vertex for the production of the particles  $P_i$  with momenta  $k_i = q_i - q_{i+1}$  in the collisions of Reggeons with momenta  $q_i$  and  $-q_{i+1}$  and color indices  $c_i$  and  $c_{i+1}$ ,  $q_0 \equiv p_A$ ,  $q_{n+1} \equiv -p_B$ ,  $s_i = (k_{i-1} + k_i)^2$ ,  $k_0 \equiv p_{\tilde{A}}$ ,  $k_{n+1} \equiv p_{\tilde{B}}$  and  $\omega_i$  stands for  $\omega(t_i)$ , with  $t_i = q_i^2$ . In the LLA,  $P_i$  can be only the state of a single gluon (see Fig. 2). By using  $s$ -channel unitarity and the

<sup>2</sup> The imaginary part gives a next-to-next-to-leading contribution in the unitarity relations.

previous expression for the production amplitudes, the amplitude of the elastic scattering process  $A + B \longrightarrow A' + B'$  at high energies can be written as

$$A_{AB}^{A'B'} = \frac{is}{(2\pi)^{D-1}} \int \frac{d^{D-2}q_1}{\vec{q}_1^2 \vec{q}_1'^2} \int \frac{d^{D-2}q_2}{\vec{q}_2^2 \vec{q}_2'^2} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{\sin(\pi\omega)} \sum_{R,\nu} \Phi_{A'A}^{(R,\nu)}(\vec{q}_1; \vec{q}; s_0) \\ \times \left[ \left( \frac{-s}{s_0} \right)^\omega - \tau \left( \frac{s}{s_0} \right)^\omega \right] G_\omega^{(R)}(\vec{q}_1, \vec{q}_2, \vec{q}) \Phi_{B'B}^{(R,\nu)}(-\vec{q}_2; -\vec{q}; s_0) . \quad (3)$$

Here and below  $q_i' \equiv q_i - q$ ,  $q \sim q_\perp$  is the momentum transfer in the process, the sum is over the irreducible representations  $R$  of the color group contained in the product of two adjoint representations and over the states  $\nu$  of these representations,  $\tau$  is the signature equal to  $+1(-1)$  for symmetric (antisymmetric) representations and  $s_0$  is an energy scale.  $\Phi_{P'P}^{(R,\nu)}$  is the so-called impact factor in the  $t$ -channel color state  $(R, \nu)$  and  $G_\omega^{(R)}$  is the Mellin transform of the Green's functions for Reggeon-Reggeon scattering (see Fig. 3). The dependence from  $s$  is determined by  $G_\omega^{(R)}$ , which obeys the equation (see Fig. 4)

$$\omega G_\omega^{(R)}(\vec{q}_1, \vec{q}_2, \vec{q}) = \vec{q}_1^2 \vec{q}_1'^2 \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) \\ + \int \frac{d^{D-2}q_r}{\vec{q}_r^2 \vec{q}_r'^2} K^{(R)}(\vec{q}_1, \vec{q}_r; \vec{q}) G_\omega^{(R)}(\vec{q}_r, \vec{q}_2; \vec{q}) , \quad (4)$$

whose integral kernel,

$$K^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) = [\omega(-\vec{q}_1^2) + \omega(-\vec{q}_1'^2)] \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + K_r^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) , \quad (5)$$

is composed by a “virtual” part, related to the gluon trajectory, and by a “real” part, related to particle production in Reggeon-Reggeon collisions. In the LLA, the “virtual” part of the kernel takes contribution from the gluon Regge trajectory with 1-loop accuracy,  $\omega^{(1)}$ , while the “real” part takes contribution from the production of one gluon in the Reggeon-Reggeon collision at Born level,  $K_{RRG}^{(B)}$ . The BFKL equation is Eq. (4) specialized for  $t = 0$  and singlet quantum numbers in the  $t$ -channel.

The representation (3) of the elastic amplitude,  $A+B \longrightarrow A'+B'$ , derived from  $s$ -channel unitarity, for the part with gluon quantum numbers in the  $t$ -channel ( $R = 8$ ,  $\tau = -1$ ), must reproduce the representation (1) with one Reggeized gluon exchange in the  $t$ -channel, with LLA accuracy. This consistency is called “bootstrap” and was checked in the LLA already in [1]. Subsequently, a rigorous proof of the gluon Reggeization in the LLA was constructed [2].

The part of the representation (3) with vacuum quantum numbers in the  $t$ -channel ( $R = 0$ ,  $\tau = +1$ ) for the case of zero momentum transfer is relevant for the total cross section of the scattering of particles  $A$  and  $B$ , via

the optical theorem. In the LLA, it turns out that  $\sigma_{AB}^{\text{LLA}} \sim s^{\omega_P^B}/\sqrt{\ln s}$ , with  $\omega_P^B = 4 \ln 2N \alpha_s/\pi$ . This relation shows that unitarity is violated, since the cross section overcomes the Froissart-Martin bound. This is obvious since, in the LLA, only a definite set of intermediate states, as we have seen, contributes to the  $s$ -channel unitarity relation. This means that the BFKL approach cannot be applied at asymptotically high energies. In order to identify the applicability region of the BFKL approach, it is necessary to know the scale of  $s$  and the argument of the running coupling constant, which are not fixed in the LLA.

### 3 BFKL in the NLA

In the NLA, the Regge form of the elastic amplitude (1) and of the production amplitudes (2), implied by gluon Reggeization, has been checked only in the first three orders of perturbation theory [4]. In order to derive the BFKL equation in the NLA, gluon Reggeization is assumed to be valid to all orders of perturbation theory. It becomes important, therefore, to check the validity of this assumption. This will be done at the end of this Section.

In the NLA it is necessary to include into the unitarity relations contributions which differ from those in the LLA by having one additional power of  $\alpha_s$  or one power less in  $\ln s$ . The first set of corrections is realized by performing, only in one place, one of the following replacements in the production amplitudes (2) entering the  $s$ -channel unitarity relation:

$$\omega^{(1)} \longrightarrow \omega^{(2)} , \quad \Gamma_{P'P}^c(\text{Born}) \longrightarrow \Gamma_{P'P}^{c(1\text{-loop})} , \quad \gamma_{c_i c_{i+1}}^{G_i(\text{Born})} \longrightarrow \gamma_{c_i c_{i+1}}^{G_i(1\text{-loop})} .$$

The second set of corrections consists in allowing the production in the  $s$ -channel intermediate state of *one* pair of particles with rapidities of the same order of magnitude, both in the central or in the fragmentation region (quasi-multi-Regge kinematics). This implies one replacement among the following in the production amplitudes (2) entering the  $s$ -channel unitarity relation:

$$\Gamma_{P'P}^c(\text{Born}) \longrightarrow \Gamma_{\{f\}P}^c(\text{Born}) , \quad \gamma_{c_i c_{i+1}}^{G_i(\text{Born})} \longrightarrow \gamma_{c_i c_{i+1}}^{Q\bar{Q}(\text{Born})} ,$$

$$\gamma_{c_i c_{i+1}}^{G_i(\text{Born})} \longrightarrow \gamma_{c_i c_{i+1}}^{GG(\text{Born})} .$$

Here  $\Gamma_{\{f\}P}$  stands for the production of a state containing an extra particle in the fragmentation region of the particle  $P$  in the scattering off the Reggeon,  $\gamma_{c_i c_{i+1}}^{Q\bar{Q}(\text{Born})}$  and  $\gamma_{c_i c_{i+1}}^{GG(\text{Born})}$  are the effective vertices for the production of a quark anti-quark pair and of a two-gluon pair, respectively, in the collision of two Reggeons.

The detailed program of next-to-leading corrections to the BFKL equation was formulated in Ref. [5]. They have been calculated over a period of several years,

mostly by one research group, lead by V.S. Fadin (for an exhaustive review, see Refs. [6]). It turns out that also in the NLA the amplitude for the high energy elastic process  $A+B \rightarrow A'+B'$  can be represented as in Eq. (3) and in Fig. 3. The Green's functions obey an equation with the same form as Eq. (4), with a kernel having the same structure as in Eq. (5). Here the “virtual” part of the kernel takes also the contribution from the gluon trajectory at 2-loop accuracy,  $\omega^{(2)}$  [4], while the “real” part of the kernel takes the additional contribution from one-gluon production in the Reggeon-Reggeon collisions at 1-loop order,  $K_{RRG}^{(1)}$  [7–10], quark anti-quark pair production at Born level,  $K_{RRQ\bar{Q}}^{(B)}$  (Ref. [11] for the forward case, Ref. [12] for the non-forward case) and from two-gluon pair at Born level,  $K_{RRGG}^{(B)}$  (Ref. [13] for the forward case, Ref. [14] for the non-forward, octet case). So, summarizing, the “virtual” part of the kernel is known in the NLA; as for the “real” part in the NLA, it is known completely for the singlet color representation in the  $t$ -channel in the forward case ( $t = 0$ ) and for the octet color representation in the  $t$ -channel in the non-forward case. The singlet NLA kernel in the non-forward case is not completely known, since the singlet  $K_{RRGG}^{(B)}$  contribution is still missing.

The consistency between the representation (3) of the elastic amplitude,  $A+B \rightarrow A'+B'$ , derived from  $s$ -channel unitarity, for the part with gluon quantum numbers in the  $t$ -channel ( $R = 8, \tau = -1$ ), and the representation (1) with one Reggeized gluon exchange in the  $t$ -channel (“bootstrap”) is of crucial importance in the NLA. In this approximation, indeed, gluon Reggeization was only assumed in order to derive the BFKL equation. Moreover, the check of the bootstrap is also a (partial) check of the correctness of calculations which, as already pointed out, were performed mostly by one research group. In the NLA, the bootstrap leads to two conditions to be verified [15], one on the NLA octet kernel, the other on the NLA octet impact factors. The first bootstrap condition has been verified at arbitrary space-time dimension for the part concerning the quark contribution to the kernel in massless QCD [12], while for the part concerning the gluon contribution to the kernel it has been verified in the  $D \rightarrow 4$  limit [16]. The second bootstrap condition is process-dependent, therefore it should be checked in principle for every octet impact factor. So far, it has been checked at arbitrary space-time dimension for quark and gluon NLA impact factors in QCD with massive quarks [17,18].

As for the exchange of vacuum quantum numbers in the  $t$ -channel, the NLA corrections to the BFKL kernel lead to a large correction to the BFKL Pomeron intercept [19,20]:  $\omega_P = \omega_P^B(1 - 2.4 \omega_P^B)$ , with  $\omega_P^B = 4 \ln 2 N \alpha_s(\vec{q}^2)/\pi$ . A lot of papers have been devoted to the problem of this large correction (see, for instance, [21]). In my opinion, the BFKL Pomeron intercept itself has not a special physical meaning. Instead, the BFKL approach can predict the full amplitude of hard QCD processes in the NLA, as soon as NLA impact factors of colorless particles accessible to perturbative QCD are known in the next-

to-leading-order. In this respect, one of the most interesting calculations, that of  $\gamma^* \rightarrow \gamma^*$  impact factors in the NLA, is in progress [22,23].

## 4 Strong bootstrap

It has been proposed by Braun and Vacca [24] that gluon Reggeization is realized also in the unphysical particle-Reggeon scattering amplitude with gluon quantum numbers in the  $t$ -channel. This requirement leads to two so-called “strong” bootstrap conditions. One of them fixes the process-dependence of the octet impact factors, stating that any octet impact factor is proportional to the corresponding effective vertex by a universal “coefficient function”. The other states that this universal coefficient function is “eigenstate” of the octet BFKL kernel, taken as operator in the transverse space, with the gluon trajectory as “eigenvalue”. The coefficient function has been determined in the NLA by the known expression of the octet quark impact factors [18] and has been used to check the strong bootstrap condition on gluon impact factors [25] in the NLA. The strong bootstrap condition on the kernel is trivially satisfied in the LLA. As for the NLA, it has been verified so far only for the quark part [24,25]. Although the physical role of the strong bootstrap conditions is not yet clear, they are of practical importance, since their fulfillment implies that of the “soft” ones [15].

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